Structurable Algebras and the Magic Square

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1 Jordan algebras

Let $\mathcal{J}$ be a (linear) *Jordan algebra* = commutative, unital nonassociative algebra in $char \neq 2$ with

$$(x^2 \cdot y) \cdot x = x^2 \cdot (y \cdot x). \tag{1}$$

Now $[L_x, L_y] \in \mathfrak{der}(\mathcal{J})$ where $L_x(y) = x \cdot y$.

Also, $[D, L_x] = L_{Dx}$ for $D \in \mathfrak{der}(\mathcal{J})$. Thus,

$$\mathfrak{inder}(\mathcal{J}) = [L_{\mathcal{J}}, L_{\mathcal{J}}]$$

is an ideal of $\mathfrak{der}(\mathcal{J})$, and

$$\mathfrak{strl}(\mathcal{J}) = L_{\mathcal{J}} \oplus \mathfrak{der}(\mathcal{J})$$

is a Lie algebra called the *structure Lie algebra* of $\mathcal{J}$ with ideal

$$\mathfrak{instrl}(\mathcal{J}) = L_{\mathcal{J}} \oplus [L_{\mathcal{J}}, L_{\mathcal{J}}].$$

Note

$$\varepsilon : L_x + D \to -L_x + D$$

is an automorphism of $\mathfrak{strl}(\mathcal{J})$.

Define

$$V_{x,y} = L_{x\cdot y} + [L_x, L_y].$$

We see $V_{x,1} = L_x$, so $V_{\mathcal{J},\mathcal{J}} = \mathfrak{instrl}(\mathcal{J})$.

Also, if $A \in \mathfrak{gl}(\mathcal{J})$, then $A \in \mathfrak{strl}(\mathcal{J}) \iff$

$$[A, V_{x,y}] = V_{Ax,y} + V_{x,By},$$

for some $B \in \mathfrak{gl}(\mathcal{J})$ and then $B = A^{\varepsilon}$.

Let $\mathcal{J}^+, \mathcal{J}^-$ be two copies of $\mathcal{J}$ and let

$$\mathfrak{instrl}(\mathcal{J}) \subset \mathcal{L} \subset \mathfrak{strl}(\mathcal{J})$$

be a subalgebra. The Tits-Kantor-Koecher Lie algebra is

$$TKK(\mathcal{J}, \mathcal{L}) = \mathcal{J}^- \oplus \mathcal{L} \oplus \mathcal{J}^+$$

with skew-symmetric product given by

$$\begin{aligned}
[\mathcal{J}^\sigma, \mathcal{J}^\sigma] &= 0, \\
[x^+, y^-] &= V_{x,y}, \\
[A, B] &= AB - BA, \\
[A, x^+] &= (Ax)^+, \\
[A, y^-] &= (A^\varepsilon y)^-
\end{aligned}$$

for $\sigma = \pm$, $x^+ \in \mathcal{J}^+$, $y^- \in \mathcal{J}^-$, $A, B \in \mathcal{L}$.

Example: $\mathcal{J} =$ Albert algebra

$=$ 27-dimensional exceptional Jordan algebra

$\mathfrak{der}(\mathcal{J})$ is a form of F_4

$\mathfrak{strl}(\mathcal{J})$ is a form of $E_6 \oplus k$

$TKK(\mathcal{J}, \mathfrak{strl}(\mathcal{J}))$ is a form of E_7

Also note $e = 1^+$, $f = 1^-$, and $h = V_{1,1}$ form a B_1-triple:

$$\begin{aligned} [e, f] &= h, \\ [h, e] &= e, \\ [h, f] &= -f. \end{aligned}$$

I.e., the multiplication is like

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \in B_1$$

rather than like the $\mathfrak{sl}_2$-triple

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathfrak{sl}_2$$

e, f, h is a B_1 triple $\iff e, 2f, 2h$ is a $\mathfrak{sl}_2$-triple

Theorem (Tits, Kantor, Koecher):Let $\mathcal{G}$ be a Lie algebra over a field of characteristic not 2 or 3 containing a B_1-triple e, f, h. If

$$\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1$$

where $\mathcal{G}_k$ is the k-eigenspace of $ad(h)$, then $\mathcal{J} = \mathcal{G}_1$ with product $x \cdot y = [[x, f], y]$ is a Jordan algebra, $\mathcal{L} = ad(\mathcal{G}_0) \mid_{\mathcal{J}}$ is a Lie algebra with

$$\mathfrak{instrl}(\mathcal{J}) \subset \mathcal{L} \subset \mathfrak{strl}(\mathcal{J}),$$

and $\mathcal{G}/I \cong TKK(\mathcal{J}, \mathcal{L})$ with $\mathcal{G}_0 \supset I \lhd \mathcal{G}$.

2 Structurable algebras

Consider a $\mathbb{Z}$-grading

$$\mathcal{G} = \mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2.$$

Kantor's approach: Given a B_1-triple, work with the *conservative product* on $\mathcal{G}_1$:

$$x \cdot y = [[x, f], y].$$

Example: $\mathcal{A}$ associative with involution $a \to \bar{a}$,

$\mathcal{S} = \{s \in \mathcal{A} : \bar{s} = -s\}$

$$\mathcal{G} = \left\{ \begin{bmatrix} c & a & s \\ b & r & -\bar{a} \\ t & -\bar{b} & -\bar{c} \end{bmatrix} : a, b, c \in \mathcal{A}, r, s, t \in \mathcal{S} \right\}$$

graded by lines parallel to the main diagonal,

so $\mathcal{G}_{\pm 1} \longleftrightarrow \mathcal{A}$ and $\mathcal{G}_{\pm 2} \longleftrightarrow \mathcal{S}$.

The conservative product on $\mathcal{A}$ is $a \cdot b = ab + ba - b\bar{a}$, while the "nice" product is ab.

Allison's approach: Make use of the involution! $\mathcal{A} =$ unital nonassociative algebra with involution $x \to \bar{x}$, $char \neq 2, 3$.

Set

$$V_{x,y}(z) = U_{x,z}(y) = \{xyz\} \\ = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y.$$

We say $(\mathcal{A}, -)$ is a *structurable algebra* if

$$[V_{x,y}, V_{z,w}] = V_{\{xyz\},w} - V_{z,\{yxw\}}. \qquad (2)$$

If the involution is trivial, then $\mathcal{A}$ is commutative, and (2) is equivalent to the Jordan identity (1),

so a Jordan algebra is just a structurable algebra with trivial involution.

As before,

$$\mathfrak{strl}(\mathcal{A}, -) = \{A : [A, V_{x,y}] = V_{Ax,y} + V_{x,By}\}$$

for some $B \in \mathfrak{gl}(\mathcal{A})$,

is a Lie algebra containing the ideal $V_{\mathcal{A},\mathcal{A}} = \mathfrak{instrl}(\mathcal{A}, -)$, $B = A^{\varepsilon}$ is determined by A,and ε is an automorphism.

Let $\mathfrak{instrl}(\mathcal{A}, -) \subset \mathcal{L} \subset \mathfrak{strl}(\mathcal{A}, -)$ be a subalgebra. Form

$$\mathcal{K}(\mathcal{A}, \mathcal{L}) = \mathcal{S} \oplus \mathcal{A} \oplus \mathcal{L} \oplus \mathcal{A} \oplus \mathcal{S}.$$

Write $(t, y, A, x, s) \in \mathcal{S} \oplus \mathcal{A} \oplus \mathcal{L} \oplus \mathcal{A} \oplus \mathcal{S} = \mathcal{K}(\mathcal{A}, \mathcal{L})$ as

$$\begin{bmatrix} A & L_s \\ L_t & A^\varepsilon \end{bmatrix} \oplus \begin{bmatrix} x \\ y \end{bmatrix} \in \mathcal{K}_{\bar{0}} \oplus \mathcal{K}_{\bar{1}}$$

and define a skew-symmetric product with

$$\begin{aligned} [C, D] &= CD - DC, \\ [C, u] &= Cu, \\ [u, v] &= u * v - v * u \end{aligned}$$

for $C, D \in \mathcal{K}_{\bar{0}}$ and $u, v \in \mathcal{K}_{\bar{1}}$ where

$$\begin{bmatrix} x \\ y \end{bmatrix} * \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} V_{x,w} & U_{x,z} \\ U_{y,w} & V_{y,z} \end{bmatrix}$$

$\mathcal{K}(\mathcal{A}, \mathcal{L})$ is $\mathbb{Z}$-graded Lie algebra and

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} Id & 0 \\ 0 & -Id \end{bmatrix}$$

is a B_1-triple.

Write $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{A}, \mathfrak{instrl}(\mathcal{A}, -))$.

Theorem (Allison): Let $\mathcal{G}$ be a Lie algebra over a field of characteristic not $2, 3$ or 5 containing a B_1-triple e, f, h. If

$$\mathcal{G}_{-2} \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_0 \oplus \mathcal{G}_1 \oplus \mathcal{G}_2$$

where $\mathcal{G}_k$ is the k-eigenspace of $ad(h)$, then $\mathcal{A} = \mathcal{G}_1$ is a structurable algebra, $\mathcal{L} = ad(\mathcal{G}_0) \mid_{\mathcal{A}}$ is a Lie algebra with

$$\mathfrak{instrl}(\mathcal{A}, -) \subset \mathcal{L} \subset \mathfrak{strl}(\mathcal{A}, -)$$

and $\mathcal{G}/I \cong \mathcal{K}(\mathcal{A}, \mathcal{L})$ with $\mathcal{G}_0 \supset I \lhd \mathcal{G}$.

3 Examples of structurable algebras

Theorem (Allison, Smirnov): Any central simple structurable algebra, $char(k) \neq 2, 3, 5$, is isomorphic to one of the following:

(a) a Jordan algebra,

(b) an associative algebra with involution,

(c) a 2×2-matrix algebra $\left[\begin{array}{cc} k & \mathcal{J} \\ \mathcal{J} & k \end{array} \right]$ constructed from the Jordan algebra $\mathcal{J}$ of an admissible cubic form with basepoint and a nonzero scalar, or a form of such an algebra, (related to Freudenthal triple systems)

(d) an algebra $\mathcal{A} \oplus \mathcal{W}$ constructed from a hermitian form on the associative $\mathcal{A}$-module $\mathcal{W}$

(e) a tensor product $(\mathcal{C}_1 \otimes \mathcal{C}_2, - \otimes -)$ of two composition algebras, or a form of such an algebra,

(f) a Kantor-Smirnov algebra $\mathcal{T}(\mathcal{C})$ constructed from an octonion algebra $\mathcal{C}$.

$\mathcal{K}(\mathcal{C}_1 \otimes \mathcal{C}_2)$ — rows: $\dim(\mathcal{C}_1)$, columns: $\dim(\mathcal{C}_2)$

	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \oplus A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

4 Another Lie algebra construction

Example: $\mathcal{A}$ associative with involution $a \longrightarrow \bar{a}$, $n \geq 3$,

$$\mathcal{G} = \{A \in \mathcal{A}_n : \bar{A}^t = -A\}$$

Set $u_{ij}(a) = ae_{ij} - \bar{a}e_{ji}$, $i \neq j$, then

$$\begin{aligned}
u_{ij}(a) &= u_{ji}(-\bar{a}), && (3)\\
a &\longrightarrow u_{ij}(a) \text{ is linear},\\
[u_{ij}(a), u_{jk}(b)] &= u_{ik}(ab) \text{ for distinct } i, j, k,\\
[u_{ij}(a), u_{kl}(b)] &= 0 \text{ for distinct } i, j, k, l.
\end{aligned}$$

Theorem (Allison & Faulkner): Let $\mathcal{A}$ be a unital nonassociative algebra with involution $x \to \bar{x}$, $char \neq 2, 3$. Let $\mathcal{G}$ be the Lie algebra generated by symbols $u_{ij}(a)$, $i \neq j$, $a \in \mathcal{A}$, subject to the relations (3). Then

$$u_{ij}(a) = 0 \Longrightarrow a = 0,$$

$\Longleftrightarrow$ either $n \geq 4$ and $\mathcal{A}$ is associative or $n = 3$ and $\mathcal{A}$ is structurable.

To prove the converse of the Theorem if $n = 3$, we use the following construction:

Let $\mathcal{A}$ be structurable. If $A \in \mathfrak{gl}(\mathcal{A})$, let $\bar{A}(x) = \overline{A(\bar{x})}$. We say $T = (T_1, T_2, T_3)$ is a *Lie-related triple* if

$$\bar{T}_i(xy) = T_j(x)y + xT_k(y)$$

for $x, y \in \mathcal{A}$, $(i, j, k) \circlearrowleft (1, 2, 3)$. These form a Lie algebra $\mathfrak{trip}(\mathcal{A})$. Given $a, b \in \mathcal{A}$ and $(i, j, k) \circlearrowleft (1, 2, 3)$, an example is

$$\begin{aligned} T_i &= L_{\bar{b}} L_a - L_{\bar{a}} L_b, \\ T_j &= R_{\bar{b}} R_a - R_{\bar{a}} R_b, \\ T_k &= R_{(\bar{a}b - \bar{b}a)} + L_b L_{\bar{a}} - L_a L_{\bar{b}} \end{aligned} \tag{4}$$

These span an ideal $\mathfrak{intrip}(\mathcal{A})$. For $i \neq j$, let $u_{ij}(\mathcal{A})$ be a copy of $\mathcal{A}$ with $u_{ij}(a) = u_{ji}(-\bar{a})$. Let

$$\mathfrak{intrip}(\mathcal{A}) \subset \mathcal{D} \subset \mathfrak{trip}(\mathcal{A})$$

be a subalgebra. Form

$$\mathcal{U}(\mathcal{A}, \mathcal{D}) = \mathcal{D} \oplus u_{12}(\mathcal{A}) \oplus u_{23}(\mathcal{A}) \oplus u_{31}(\mathcal{A})$$

with skew-symmetric product given by

$$\begin{aligned} [u_{ij}(a), u_{jk}(b)] &= u_{ik}(ab) \text{ for distinct } i, j, k, \\ [u_{ij}(a), u_{ij}(b)] &= T \text{ as in (4)}, \\ [T, u_{ij}(a)] &= u_{ij}(T_k(a)) \text{ for } (i, j, k) \circlearrowleft (1, 2, 3). \end{aligned}$$

$\mathcal{U}(\mathcal{A}, \mathcal{D})$ is a Lie algebra. Moreover, if the basefield is algebraically closed, then

$$\mathcal{U}(\mathcal{A}) := \mathcal{U}(\mathcal{A}, \mathfrak{intrip}(\mathcal{A})) \cong \mathcal{K}(\mathcal{A}).$$

In particular, $\mathcal{U}(\mathcal{C}_1 \otimes \mathcal{C}_2)$ gives the magic square.

Let K be the Klein 4-group with $\kappa_i = (jk)(i4)$ for $\{i, j, k\} = \{1, 2, 3\}$. Suppose S_4 acts on a vector space $\mathcal{V}$. We have

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3,$$

where

$$\begin{aligned} \mathcal{V}_0 &= \{v \in \mathcal{V} : \kappa(x) = x, \text{ for } \kappa \in K\}, \\ \mathcal{V}_i &= \{v \in \mathcal{V} : \kappa_j(x) = -x, \text{ for } j \neq i\}. \end{aligned}$$

Note $S_4 = S_3 \ltimes K$, so S_3 permutes $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$.

Theorem (Elduque & Okubo): Let S_4 act by automorphisms on a Lie algebra $\mathcal{G}$. If $u \in \mathcal{G}_3$ with

$$(12)(a) = [u, a]$$

for all $a \in \mathcal{G}_1$, then $\mathcal{A} = \mathcal{G}_3$ is a structurable algebra with product

$$ab = [(23)a, (31)b],$$

involution $\bar{a} = -(12)a$ and identity u. Moreover, if

$$\mathcal{G}_0' = ad(\mathcal{G}_0) \mid_{\mathcal{G}_1 \oplus \mathcal{G}_2 \oplus \mathcal{G}_3},$$

then

$$\mathfrak{intrip}(\mathcal{A}) \subset \mathcal{G}_0' \subset \mathfrak{trip}(\mathcal{A})$$

and $\mathcal{G}/I \cong \mathcal{U}(\mathcal{A}, \mathcal{G}_0')$ with $\mathcal{G}_0 \supset I \lhd \mathcal{G}$.

Idea of proof: Set $u_{ij}(a) = \sigma(a)$ for $a \in \mathcal{A} = \mathcal{G}_3$ where $\sigma \in S_3$ with $\sigma : 1, 2 \to i, j$. Show $u_{ij}(a)$ satisfies (3).

5 Symmetric spaces

Following Loos, we recall that a *symmetric space* is a manifold M with a product $M \times M \to M$, (written as $S_x y = x \cdot y$ and called *reflection*) such that

$$S_x^2 = id,$$
$$S_x S_y S_x = S_{S_x y},$$
$$x \text{ is an isolated fixed point of } S_x.$$

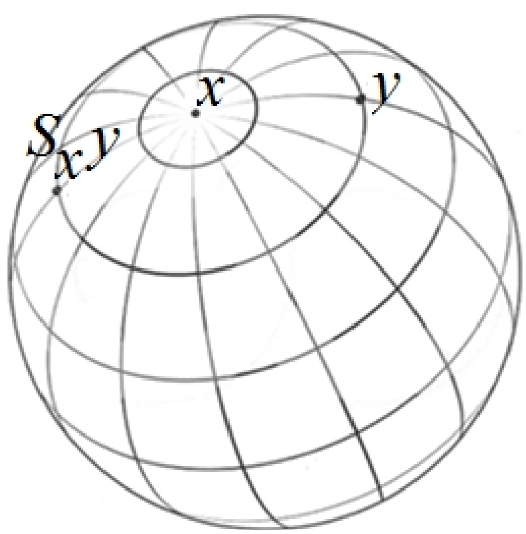
The group $G(M)$ generated by all $S_x S_y$ is the *group of displacements*.

Let $\mathcal{G}(M)$ be the Lie algebra of $G(M)$. S_x induces an automorphism S_x of $\mathcal{G}(M)$.

The tangent space at x can be identified with the (-1)-eigenspace $\mathcal{G}(M)_{-1}$ of S_x.

This is a Lie triple system (closed under $[[,],]$) and is the negative of the curvature tensor of the canonical connection.

Example 1: $M = S^2$ is the unit sphere and S_x is rotation by 180° about x.



In this case,

$$\begin{aligned} G(M) &= SO_3(\mathbb{R}), \mathcal{G}(M) = \mathfrak{so}_3(\mathbb{R}), \\ T_{e_3}(M) &= \mathfrak{so}_3(\mathbb{R})_{-1} = u_{13}(\mathbb{R}) \oplus u_{23}(\mathbb{R}). \end{aligned}$$

Example 2: $M = E^2$ is the real projective (elliptic) plane; i.e., the sphere with antipodal points identified. S_x, $G(M)$, etc. are the same.

Example 3: G a connected Lie group with an automorphism σ of order 2. Let G^σ be the fixed points of σ and $G_0^\sigma \subset K \subset G^\sigma$. Let $M = G/K$ with

$$S_{xK} = L_x \sigma L_{x^{-1}}.$$

We say that a symmetric space M is *rotational* if

(1) M has a symmetric subspace $N \cong S^2$ or E^2,

(2) if $x, y, z \in N$ with $y, z \perp x$ and $S_y \neq S_z$, then x is an isolated fixed point of $S_y S_z$.

Theorem (Faulkner): A connected symmetric space is rotational $\Longleftrightarrow$ its Lie triple system is isomorphic to

$$u_{13}(\mathcal{A}) \oplus u_{23}(\mathcal{A})$$

for some real structurable algebra $\mathcal{A}$.

Note: locally the symmetric space has coordinates $\{(a, b) : a, b \in \mathcal{A}\}$

Idea of Proof: The subgroup $\tilde{G}(N)$ of $G(M)$ generated by all $S_x S_y$, $x, y \in N$ is isomorphic to $G(N) = SO_3(\mathbb{R})$. Thus, $SO_3(\mathbb{R})$ acts as automorphisms of $\mathcal{G}(M)$. The subgroup of rotations of the cube is isomorphic to S_4 (acting on the 4 diagonals).

$\mathcal{U}(\mathcal{C}_1 \otimes \mathcal{C}_2)/Lie(K)$ with $\dim(\mathcal{C}_1) = 8$ is given by

$\dim(\mathcal{C}_2)$

1	2	4	8
F_4/B_4	$E_6/(D_5 \oplus \mathbb{R})$	$E_7/(D_6 \oplus A_1)$	E_8/D_8

6 Kantor-Smirnov algebras

Let $\mathcal{C}$ be a composition algebra. Let τ be the automorphism of $\mathcal{C} \otimes \mathcal{C}$ with $\tau : a \otimes b \longrightarrow b \otimes a$. Let $(\mathcal{C} \otimes \mathcal{C})^\tau$ be the subalgebra of fixed points of τ.

We find an ideal of dimension 1 in $(\mathcal{C} \otimes \mathcal{C})^\tau$ as follows:

It is easy to check that the linear map

$$\varphi : \mathcal{C} \otimes \mathcal{C} \to End(\mathcal{C})$$

with

$$\varphi(a \otimes b)(x) = an(b, x)$$

is an isomorphism of $(\mathcal{C} \otimes \mathcal{C})$-bimodules with involution, where

$$\begin{aligned} (a \otimes b) \cdot A &= L_a A L_{\bar{b}}, \\ A \cdot (a \otimes b) &= R_a A R_{\bar{b}}, \\ \bar{A}(x) &= \overline{A(\bar{x})}. \end{aligned}$$

Moreover, $* \circ \varphi = \varphi \circ \tau$ where $n(Ax, y) = n(x, A^* y)$. Thus,

$$\varphi : (\mathcal{C} \otimes \mathcal{C})^\tau \to \mathcal{H}(End(\mathcal{C})) = \{A \in End(\mathcal{C}) : A^* = A\}$$

is an isomorphism of $(\mathcal{C} \otimes \mathcal{C})^\tau$-bimodules with involution.

Since

$$\begin{aligned} (a \otimes a) \cdot Id &= L_a L_{\bar{a}} = n(a) Id = Id \cdot (a \otimes a), \\ \overline{Id} &= Id, \end{aligned}$$

kId is a submodule of $\mathcal{H}(End(\mathcal{C}))$

and $\varphi^{-1}(kId)$ is an ideal of $(\mathcal{C} \otimes \mathcal{C})^\tau$.

Let $\mathcal{T}(\mathcal{C}) = (\mathcal{C} \otimes \mathcal{C})^\tau / \varphi^{-1}(kId)$.

$\dim \mathcal{C}$	$\mathcal{T}(\mathcal{C})$	$\mathcal{U}(\mathcal{T}(\mathcal{C}))$	$Lie(K)$
1	0	0	0
2	$\cong (\mathcal{C}, -)$	A_2	$\mathbb{R} \oplus A_1$
4	$\cong (End(\mathcal{S}(\mathcal{C})), *)$	B_4	$A_1 \oplus A_2$
8	Kantor-Smirnov	E_7	A_7